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Characterization of 1-greedy bases☆

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Abstract

A basis for a Banach space X is greedy if and only if the greedy algorithm provides, up to a constant C depending only on X, the best m-term approximation for each element of the space. It is known that the Haar (or good wavelet) basis is a greedy basis in $L_p(0, 1)$ for $1 < p < \infty$ [V.N. Temlyakov, The best m-term approximation and greedy algorithms, Adv. in Comp. Math. 8 (1998) 249–265]. In this particular example, unfortunately, the constant of greediness $C = C(p)$ is strictly bigger than 1 unless $p = 2$. Our goal is to investigate 1-greedy bases, i.e., bases for which the greedy algorithm provides the best m-term approximation. We find a characterization of 1-greediness, study how 1-greedy bases relate to symmetric bases, and show that 1-greediness does not imply 1-symmetry, answering thus two questions raised in [P. Wojtaszczyk, Greedy Type Bases in Banach Spaces, Constructive Function Theory, Varna 2002, Darba, Sofia, 2002, pp. 1–20].

1. Introduction

Let X be a (real) Banach space with a semi-normalized basis $(e_n)_{n=1}^{\eta}$ (η finite or infinite). For each $m = 1, 2, \ldots$, we let Σ_m denote the collection of all elements of X which can be expressed

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as a linear combination of m elements of (e_n):

$$\Sigma_m = \left\{ y = \sum_{j \in B} \alpha_j e_j \, : \, B \subset \mathbb{N}, |B| = m, \alpha_j\text{'s scalars} \right\}.$$

Let us note that the space Σ_m is not linear: the sum of two elements from Σ_m is generally not in Σ_m, it is in Σ_{2m}. An approximation algorithm is a sequence of maps $T_m : X \to X$, $m \in \mathbb{N}$, so that for each $x \in X$, $T_m(x) \in \Sigma_m$. For $x \in X$, its *best m-term approximation error* (with respect to the given basis) is

$$\sigma_m(x) = \inf_{y \in \Sigma_m} \|x - y\|.$$

The fundamental question is how to construct an approximation algorithm which for every $x \in X$ and each m produces an element $T_m(x) \in \Sigma_m$ so that the error of the approximation of x by $T_m(x)$ be (uniformly) comparable with $\sigma_m(x)$, i.e.

$$\|x - T_m(x)\| \leqslant C\sigma_m(x),$$

where C is an absolute constant.

The most obvious and in some sense natural attempt to get such an algorithm is to consider the *Greedy Algorithm*, $(\mathcal{G}_m)_{m=1}^{\eta}$, where for each x, $\mathcal{G}_m(x)$ is obtained by taking the largest m coefficients in the series expansion of x. To be precise, if we let $(e_n^*)_{n=1}^{\eta} \subset X^*$ denote the biorthogonal functionals associated to $(e_n)_{n=1}^{\eta}$, for $x \in X$ put

$$\mathcal{G}_m(x) = \sum_{j \in B} e_j^*(x) e_j,$$

where the set $B \subset \mathbb{N}$ is chosen in such a way that $|B| = m$ and $|e_j^*(x)| \geqslant |e_k^*(x)|$ whenever $j \in B$ and $k \notin B$.

Let us note that it may happen that for some x and m the set B, hence the element $\mathcal{G}_m(x)$, is not uniquely determined by the previous conditions. In such a case, we pick any of them. Besides, the maps $\mathcal{G}_m$ are neither linear (even when the sets B are uniquely determined) nor continuous.

Following [1], given $x \in X$ we define its *greedy ordering* as the map $\rho : \{1, 2, \ldots, \eta\} \to \{1, 2, \ldots, \eta\}$ such that $\{j : e_j^*(x) \neq 0\} \subset \rho\Big(\{1, 2, \ldots, \eta\}\Big)$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. With this notation, the mth greedy approximation of x is now uniquely determined by

$$\mathcal{G}_m(x) = \sum_{j=1}^{m} e_{\rho(j)}^*(x) e_{\rho(j)}.$$

Konyagin and Temlyakov [6] defined a basis to be *C-greedy* ($C \geqslant 1$) if for all $x \in X$ and $m \in \mathbb{N}$, we have

$$\|x - \mathcal{G}_m(x)\| \leqslant C\sigma_m(x). \tag{1}$$

The smallest such constant C is the *greedy constant* of (e_n).

Note that if $C = 1$ in Eq. (1) then $\|x - \mathcal{G}_m(x)\| = \sigma_m(x)$ for all $x \in X$ and $m = 1, 2, \dots$, so the greedy algorithm gives the *best* m-term approximation for each $x \in X$.

They also defined a basis $(e_n)_{n=1}^{\eta}$ to be Δ-*democratic* ($\Delta \geqslant 1$) if for any two finite subsets A, B of $\mathbb{N}$ with $|A| = |B|$ we have

$$\left\| \sum_{k\in A} e_k \right\| \leqslant \Delta \left\| \sum_{k\in B} e_k \right\|,$$

(the least such constant Δ is the *democratic constant* of $(e_n)_{n=1}^{\eta}$) and gave the following characterization of greedy bases:

Theorem 1.1 (*Konyagin and Temlyakov [6, Theorem 1]; cf. Wojtaszczyk [11, Theorem 1]*). *If $(e_n)_{n=1}^{\eta}$ is a greedy basis with greedy constant $\leqslant C$, the (e_n) is unconditional with suppression constant $\leqslant C$ and democratic with democratic constant $\leqslant C$. Conversely, if (e_n) is unconditional with suppression constant K_s and Δ-democratic then (e_n) is greedy with greedy constant $\leqslant K_s + K_s^3\Delta$.*

We will remind the reader the notion of unconditional basis in the next Section. If we disregard constants, Theorem 1.1 says that a basis is greedy if and only if it is unconditional and democratic. In particular, Theorem 1.1 immediately yields that a 1-greedy basis has both suppression constant and democratic constant equal to 1. However, this is not a characterization of bases with greedy constant 1. In this paper we tackle the problem of finding a characterization for greedy bases with greedy constant equal to 1.

In §2, we pay close attention to the unconditional constants of an unconditional basis in relation to Theorem 1.1 and show that 1-symmetric bases are 1-greedy.

In §3, we introduce a weak symmetry condition for bases that 1-greedy bases enjoy, which we called property (A), and characterize 1-greedy bases in terms of Property A and unconditionality.

§4 deals with the problem of renorming equivalently a given Banach space X with a greedy (respectively unconditional or/and democratic) basis (e_n) in such a way that after renorming we improve the greedy constant of (e_n) (respectively, its constant of unconditionality or/and its democratic constant). We prove that for all $\varepsilon > 0$ we can extract a large "lacunary subbasis" of the Haar system in $L_p[0, 1]$ which is 1-unconditional and $(1 + \varepsilon)$-democratic.

From Theorem 1.1 it is immediate to see that greediness is a stepping stone from symmetry to unconditionality. Motivated by finding out more about the converse path, in §5 we provide examples (some of which are non-trivial) that distinguish between a variety of closely related properties of bases in Banach spaces. In particular, we see that a 1-greedy basis need not be 1-symmetric.

Finally, in §6 we give a list of open problems that arise naturally from this article.

We use standard Banach space notation and terminology throughout (see e.g. [9,7]). For clarity, however, we single out the following. $|\cdot|$ may denote (depending on the context) either the absolute value of a real number or the cardinality of a finite set. The *convex hull* of a set S (i.e., the set of all convex combinations of points of S) will be denoted by $\mathrm{co}(S)$. Given a sequence $(x_n)_{n=1}^{\eta}$ in X, we say that $(x_n)_{n=1}^{\eta}$ is *semi-normalized* (respectively, *normalized*) if there exists a constant $c > 0$ so that $1/c \leqslant \|x_n\| \leqslant c$ (respectively, $\|x_n\| = 1$) for all n. The closed linear span of $(x_n)_{n=1}^{\eta}$ is denoted by $[x_n]$. c_{00} will denote the sequence space consisting of sequences with only finitely many nonzero terms. Other concepts from the theory of bases will be introduced as needed.

2. Preliminary results

To begin let us recall that a basis $(e_n)_{n=1}^{\eta}$ of a Banach space X is said to be K-*unconditional* ($K \geqslant 1$) if for any $N \in \mathbb{N}$, whenever $a_1, \dots, a_N, b_1, \dots, b_N$ are scalars satisfying $|a_n| \leqslant |b_n|$ for $n = 1, \dots, N$, then the following inequality holds

$$\left\| \sum_{n=1}^{N} a_n e_n \right\| \leqslant K \left\| \sum_{n=1}^{N} b_n e_n \right\|. \tag{2}$$

The *unconditional constant* K_u of (e_n) is the least such constant K.

If $(e_n)_{n=1}^{\eta}$ is an unconditional basis of X and A is a subset of the integers then there is a bounded linear projection P_A from X onto $[e_k : k \in A]$ defined for each $x = \sum_{k=1}^{\eta} e_k^*(x)e_k$ by

$$P_A(x) = \sum_{k \in A} e_k^*(x)e_k.$$

$\{P_A; A \subset \mathbb{N}\}$ are the natural projections associated to the unconditional basis (e_n), and the quantity

$$K_s = \sup_A \|P_A\| < \infty$$

is called the *suppression constant* of the basis.

Let us observe that in general we have $1 \leqslant K_s \leqslant K_u \leqslant 2K_s$ (see, for instance, [7, p. 380]), but there is a situation in which K_s plays the role of K_u in Eq. (2):

Proposition 2.1. *Let $(e_n)_{n=1}^{\eta}$ be an unconditional basis for a Banach space X. Assume $a_1, \dots, a_N, b_1, \dots, b_N$ are scalars so that $|a_n| \leqslant |b_n|$ for all $1 \leqslant n \leqslant N$ and, moreover,* $\operatorname{sgn}(a_n) = \operatorname{sgn}(b_n)$ *whenever $a_n b_n \neq 0$. Then*

$$\left\| \sum_{n=1}^{N} a_n e_n \right\| \leqslant K_s \left\| \sum_{n=1}^{N} b_n e_n \right\|.$$

Proof. Fix any $N \in \mathbb{N}$ and let $a_1, \dots, a_N, b_1, \dots, b_N$ be scalars as in the hypothesis. Observe that for each $1 \leqslant n \leqslant N$ we have

$$\frac{a_n}{b_n} = \int_0^{\frac{a_n}{b_n}} 1\, dt,$$

so that we can write

$$\sum_{n=1}^{N} a_n e_n = \sum_{n=1}^{N} \int_0^1 b_n \chi_{(0, \frac{a_n}{b_n})}(t)\, dt\, e_n = \int_0^1 \left(\sum_{n=1}^{N} b_n \chi_{(0, \frac{a_n}{b_n})}(t)\, dt\, e_n \right) dt.$$

Note that for each $t \in (0, 1)$, the unconditionality of the basis yields

$$\left\| \sum_{n=1}^{N} b_n \chi_{(0, \frac{a_n}{b_n})}(t) e_n \right\| \leqslant K_s \left\| \sum_{n=1}^{N} b_n e_n \right\|. \tag{3}$$

Then, combining the properties of the Bochner integral with Eq. (3), we obtain

$$\left\|\sum_{n=1}^{N} a_n e_n\right\| \leqslant \int_0^1 \left\|\sum_{n=1}^{N} b_n \chi_{(0,\frac{a_n}{b_n})}(t) e_n\right\| dt \leqslant K_s \left\|\sum_{n=1}^{N} b_n e_n\right\|. \qquad \square$$

A basis $(e_n)_{n=1}^{\eta}$ is said to be Γ-*superdemocratic* ($\Gamma \geqslant 1$) [6] if the inequality

$$\left\|\sum_{k\in P} \theta_k e_k\right\| \leqslant \Gamma \left\|\sum_{k\in Q} \varepsilon_k e_k\right\|$$

holds for any two finite sets of integers P and Q of the same cardinality, and any choices of signs $(\theta_k)_{k\in P}$ and $(\varepsilon_k)_{k\in Q}$.

It is clear that if a basis $(e_n)_{n=1}^{\eta}$ is simultaneously K-unconditional and Δ-democratic then it is Γ-superdemocratic with $\Gamma \leqslant K^2\Delta$.

Now we show a very simple fact that we will need later.

Proposition 2.2. *Let X be a 2-dimensional Banach space with normalized basis (e_1, e_2). If (e_1, e_2) is unconditional with $K_s = 1$ then (e_1, e_2) is 1-greedy.*

Proof. We need only show that for each $x \in X$ we have

$$\|x - \mathcal{G}_1(x)\| \leqslant \sigma_1(x).$$

Put $x = \alpha e_1 + \beta e_2$. Clearly we have

$$\sigma_1(x) = \inf_{s,t} \left\{ \|(\alpha - s)e_1 + \beta e_2\|, \|\alpha e_1 + (\beta - t)e_2\| \right\}.$$

Without loss of generality we assume that $|\alpha| \geqslant |\beta|$. Using the hypothesis we obtain,

$$\|x - \mathcal{G}_1(x)\| = \|\beta e_2\| = \left\| P_{\{2\}}\Big((\alpha - s)e_1 + \beta e_2\Big)\right\| \leqslant \|(\alpha - s)e_1 + \beta e_2\|$$

and

$$\|x - \mathcal{G}_1(x)\| = |\beta| \leqslant |\alpha| = \|\alpha e_1\| = \left\| P_{\{1\}}\Big(\alpha e_1 + (\beta - t)e_2\Big)\right\| \leqslant \|\alpha e_1 + (\beta - t)e_2\|.$$

Thus, $\|x - \mathcal{G}_1(x)\| \leqslant \sigma_1(x)$ and we are done. $\square$

There are weaker forms of greediness. For any basis $(e_n)_{n=1}^{\eta}$, let

$$\tilde{\sigma}_m(x) = \inf \left\{ \left\| x - \sum_{k\in A} e_k^*(x) e_k \right\| \ : \ A \subset \{1, 2, \ldots, \eta\}, |A| \leqslant m \right\}. \tag{4}$$

A basis $(e_n)_{n=1}^{\eta}$ is *almost greedy* [1] if there is a constant C so that for each $x \in X$ and $m = 1, 2, \ldots$ we have

$$\|x - \mathcal{G}_m(x)\| \leqslant C\tilde{\sigma}_m(x).$$

A basis $(e_n)_{n=1}^{\eta}$ is *quasi-greedy* [6] if for each $x \in X$ the norm limit $\lim_{m\to\infty} \mathcal{G}_m(x)$ exists and equals x. This is equivalent (see [10]) to the condition that for some constant C

$$\sup_m \|\mathcal{G}_m(x)\| \leqslant C\|x\|.$$

Obviously,

$$\sigma_m(x) \leqslant \tilde{\sigma}_m(x) \leqslant \left\| x - \sum_{k=1}^{m} e_k^*(x)e_k \right\| \to 0 \quad \text{as } m \to \infty.$$

The following result appeared in [11]:

Proposition 2.3 (*cf. Wojtaszczyk [10, Proposition 7]*). *Let $(e_n)_{n=1}^{\eta}$ be an unconditional basis for a Banach space X with $K_s = 1$. Then, for each $x \in X$ and each $m = 1, 2, \dots$, there exists $B \subset \mathbb{N}$ of cardinality m such that*

$$\sigma_m(x) = \left\| x - \sum_{n\in B} e_n^*(x)e_n \right\|.$$

That is, if $K_s = 1$ then $\sigma_m(x) = \tilde{\sigma}_m(x)$ and the infimum in Eq. (4) is attained. Therefore, we obtain the following immediate consequence that we state for reference.

Proposition 2.4. *Let $(e_n)_{n=1}^{\eta}$ be a basis of a Banach space X.*

(i) *If (e_n) is 1-greedy, then*

$$\|x - \mathcal{G}_m(x)\| = \sigma_m(x) = \min\left\{ \left\| x - \sum_{k\in A} e_k^*(x)e_k \right\| : A \subset \{1, 2, \dots, \eta\}, |A| = m \right\}.$$

(ii) *If (e_n) is unconditional with $K_s = 1$ and*

$$\|x - \mathcal{G}_m(x)\| = \min\left\{ \left\| x - \sum_{k\in A} e_k^*(x)e_k \right\| \ : \ A \subset \{1, 2, \dots, \eta\}, |A| = m \right\}.$$

for each $x \in X$ and every $1 \leqslant m < \eta$, then (e_n) is 1-greedy.

Let us recall that an unconditional basis $(e_n)_{n=1}^{\eta}$ of a Banach space X is *symmetric* if for any permutation σ of $\{1, 2, \dots, \eta\}$, the basis $(e_{\sigma(n)})_{n=1}^{\eta}$ is *equivalent* to $(e_n)_{n=1}^{\eta}$, i.e., there is a constant C so that for any permutation σ and any choice of scalars $(a_k) \in c_{00}$ we have

$$C^{-1}\left\| \sum_{n=1}^{\eta} a_n e_n \right\| \leqslant \left\| \sum_{n=1}^{\eta} a_n e_{\sigma(n)} \right\| \leqslant C \left\| \sum_{n=1}^{\eta} a_n e_n \right\|.$$

$(e_n)_{n=1}^{\eta}$ is *K-symmetric* if for all $x = \sum_{n=1}^{\eta} a_n e_n$ the inequality

$$\left\| \sum_{n=1}^{\eta} \varepsilon_n a_n e_{\sigma(n)} \right\| \leqslant K \left\| \sum_{n=1}^{\eta} a_n e_n \right\|$$

holds for any sequence of signs (ε_n) and any permutation σ. The least such constant K is called the *symmetric constant* of $(e_n)_{n=1}^{\eta}$.

A 1-symmetric basis is, in particular, 1-unconditional and 1-democratic. Therefore, by Theorem 1.1, a 1-symmetric basis is greedy with greedy constant $\leqslant 2$. Actually, more can be said:

Theorem 2.5. *If* $(e_n)_{n=1}^{\eta}$ *is* 1*-symmetric, then* $(e_n)_{n=1}^{\eta}$ *is* 1*-greedy.*

Proof. Fix $x = \sum_{n=1}^{\eta} e_n^*(x)e_n$ and $1 \leqslant m < \eta$. Let ρ be the greedy ordering for x and $A = \{\rho(1), \rho(2), \dots, \rho(m)\}$. Thus, $\mathcal{G}_m(x) = \sum_{n\in A} e_n^*(x)e_n$. We aim to show that

$$\|x - \mathcal{G}_m(x)\| = \min\left\{\left\|x - \sum_{n\in B} e_n^*(x)e_n\right\| \ : \ B \subset \mathbb{N},\ |B| = m\right\}.$$

Given $B \subset \mathbb{N}$ of cardinality m, suppose $A \cap B = \emptyset$. If we take any permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(A) = B$ and $\pi(n) = n$ if $n \notin A \cup B$, using the 1-symmetry of the basis we have

$$\begin{aligned}\left\|x - \sum_{n\in B} e_n^*(x)e_n\right\| &= \left\|\sum_{n\in A} e_n^*(x)e_n + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &= \left\|\sum_{n\in A} e_n^*(x)e_{\pi(n)} + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &\geqslant \left\|\sum_{n\in B} e_n^*(x)e_n + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &= \left\|x - \sum_{n\in A} e_n^*(x)e_n\right\|.\end{aligned}$$

Let us assume now that $A \cap B \neq \emptyset$. We pick a permutation $\pi : \mathbb{N} \to \mathbb{N}$ so that $\pi(A\backslash B) = B\backslash A$ and $\pi(j) = j$ if $j \notin (A\backslash B \ \mathrm{sup}(B\backslash A)$. Then, the 1-symmetry of the basis yields

$$\begin{aligned}\left\|x - \sum_{n\in B} e_n^*(x)e_n\right\| &= \left\|\sum_{n\in A\backslash B} e_n^*(x)e_n + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &= \left\|\sum_{n\in A\backslash B} e_n^*(x)e_{\pi(n)} + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &\geqslant \left\|\sum_{n\in B\backslash A} e_n^*(x)e_n + \sum_{n\notin A\cup B} e_n^*(x)e_n\right\| \\ &= \left\|\sum_{n\notin A} e_n^*(x)e_n\right\| = \left\|x - \mathcal{G}_m(x)\right\|. \qquad \square\end{aligned}$$

3. Property (A)

Let $(e_n)_{n=1}^{\eta}$ be a basis of a Banach space X. Given any $x = \sum_{n=1}^{\eta} e_n^*(x)e_n \in X$, the *support of* x, denoted $\operatorname{supp} x$, consists of those n such that $e_n^*(x) \neq 0$. Let $M(x)$ denote the subset of

$\operatorname{supp} x$, where the coordinates of x (in absolute value) are the largest. Obviously the cardinality of $M(x)$ is finite for all $x \in X$. We will say that a $1-1$ map $\pi :\ \operatorname{supp} x \to \{1, 2, \dots, \eta\}$ is a *greedy permutation for* x if $\pi(j) = j$ for all $j \in \operatorname{supp} x \setminus M(x)$ and if $j \in M(x)$ then, either $\pi(j) = j$ or $\pi(j) \in \mathbb{N} \setminus \operatorname{supp} x$. That is, a greedy permutation of x puts those coefficients of x whose absolute value is the largest (or some of them) in "gaps" of the support of x, if there are any. If $\operatorname{supp} x \neq \mathbb{N}$, we will put $M^*_\pi(x) = \{j \in M(x) :\ \pi(j) \neq j\}$. $\Pi_G(x)$ will denote the set of all greedy permutations of x.

Definition 3.1. A basis $(e_n)_{n=1}^{\eta}$ for a Banach space X has property (A) if for any $x \in X$ we have

$$\left\| \sum_{n \in \operatorname{supp} x} e^*_n(x) e_n \right\| = \left\| \sum_{n \in \operatorname{supp} x} \theta_{\pi(n)} e^*_n(x) e_{\pi(n)} \right\|$$

for all $\pi \in \Pi_G(x)$ and all signs (θ_k) with $\theta_{\pi(n)} = 1$ if $n \notin M^*_\pi(x)$.

Roughly speaking, property (A) is a weak symmetry condition for largest coefficients. It allows some symmetry in the norm of a vector provided its support has "gaps". When $\operatorname{supp} x = \{1, 2, \dots, \eta\}$, then $\Pi_G(x)$ consists only of the identity permutation and the basis does not allow any symmetry in the norm of x. The opposite extreme case occurs when $x = \alpha \sum_{n \in S} e_k$, with $|\operatorname{supp} x| < \eta$; then $\|x\| = \|\alpha \sum_{k \in P} e_k\|$ for any $P \subset \{1, 2, \dots, \eta\}$ of cardinality $|\operatorname{supp} x|$. In particular, if a basis $(e_n)_{n=1}^{\eta}$ satisfies property (A) then it is 1-democratic. In fact, we have:

Proposition 3.2. *Let* $(e_n)_{n=1}^{\infty}$ *be a basis of a Banach space* X. *If* $(e_n)_{n=1}^{\infty}$ *has property* (A) *then* $(e_n)_{n=1}^{\infty}$ *is* 1-*superdemocratic.*

Proof. Given $m \in \mathbb{N}$, let A and B be any two subsets of $\mathbb{N}$ of cardinality m. We want to prove that for any choice of signs (ε_k) and (θ_k) we have

$$\left\| \sum_{k \in A} \varepsilon_k e_k \right\| = \left\| \sum_{k \in B} \theta_k e_k \right\|.$$

But, if we pick a subset C of integers of cardinality m which is disjoint with both A and B, using property (A) twice we obtain

$$\left\| \sum_{k \in A} \varepsilon_k e_k \right\| = \left\| \sum_{k \in C} e_k \right\| = \left\| \sum_{k \in B} \theta_k e_k \right\|. \qquad \square$$

Example 3.3. Let $(H_n^{(p)})_{n=1}^{\infty}$ be the Haar system normalized in $L_p[0, 1]$ for $1 \leqslant p < \infty$: $H_1^{(p)} = 1$ on $[0, 1]$ and for $n = 2^k + s$, $k = 0, 1, 2, \dots$, $s = 1, 2, \dots, 2^k$,

$$H_n^{(p)}(t) = \begin{cases} 2^{k/p} & \text{if } t \in [\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}}), \\ -2^{k/p} & \text{if } t \in [\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}}), \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\|H_1^{(p)} + 2^{-1/p} H_3^{(p)}\|_p^p \neq \|H_4^{(p)} + 2^{-1/p} H_3^{(p)}\|_p^p,$$

the Haar system does not have property (A).

Now we come to the main result of this section:

Theorem 3.4. *A basis $(e_n)_{n=1}^{\eta}$ for a Banach space X is* 1*-greedy if and only if (e_n) is unconditional with $K_s = 1$ and satisfies property* (A).

Proof. If (e_n) is 1-greedy then $K_s = 1$ by Theorem 1.1. To see that (e_n) has property (A), fix $x \in X$ and assume that $S = \operatorname{supp} x$ is a proper subset of $\{1, 2, \dots, \eta\}$, otherwise there is nothing to prove. Given π, a greedy permutation of x, and a choice of signs $\theta = (\theta_k)$ such that $\theta_{\pi(n)} = 1$ if $n \notin M_\pi^*(x)$, put $x_{\theta,\pi} = \sum_{n\in S} \theta_{\pi(n)} a_n e_{\pi(n)}$. We want to show that $\|x\| = \|x_{\theta,\pi}\|$. Consider the vector

$$y = x + \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)},$$

which results from putting as many largest coefficients of x (possibly with different signs) as $|M_\pi^*(x)|$ in gaps of the support of x. Then, on the one hand, if $m = |M_\pi^*(x)|$ we have

$$\mathcal{G}_m(y) = \sum_{k\in M_\pi^*(x)} a_k e_k.$$

Since (e_n) is 1-greedy,

$$\|x_{\theta,\pi}\| = \|y - \mathcal{G}_m(y)\| = \sigma_m(y) \leqslant \left\| y - \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)} \right\| = \|x\|.$$

On the other hand we also have

$$\mathcal{G}_m(y) = \sum_{k\in M_\pi^*(x)} \theta_{\pi(k)} a_k e_{\pi(k)},$$

hence

$$\|x\| = \|y - \mathcal{G}_m(y)\| \leqslant \left\| y - \sum_{k\in M_\pi^*(x)} a_k e_k \right\| = \|x_{\theta,\pi}\|.$$

For the converse, since $K_s = 1$, using Proposition 2.4, we will prove that (e_n) is 1-greedy by showing that for each $m \in \mathbb{N}$, $m < \eta$, and any $x \in X$, we have

$$\|x - \mathcal{G}_m(x)\| = \min\left\{ \left\| x - P_B(x) \right\| : B \subset \{1, 2, \dots, \eta\}, |B| = m \right\}.$$

Let ρ be the greedy ordering for x and $A = \{\rho(1), \rho(2), \dots, \rho(m)\}$. Thus, $\mathcal{G}_m(x) = \sum_{n\in A} e_n^*(x) e_n$. Suppose, first, that B is disjoint with A. Then, if we pick signs $(\theta_n)_{n\in A}$ so that

$\operatorname{sgn}(\theta_n e^*_{\rho(m)}(x)) = \operatorname{sgn} e^*_n(x)$ for all $n \in A$, using Proposition 2.1 we obtain

$$\begin{aligned}\left\|x - P_B(x)\right\| &= \left\|\sum_{n\in A} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &\geqslant \left\|\sum_{n\in A} \theta_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\|.\end{aligned}$$

Now pick signs $(\varepsilon_n)_{n\in B}$ so that $\operatorname{sgn}(\varepsilon_n e^*_{\rho(m)}(x)) = \operatorname{sgn} e^*_n(x)$ for each $n \in B$. Then property (A) gives

$$\left\|\sum_{n\in A} \theta_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \geqslant \left\|\sum_{n\in B} \varepsilon_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\|,$$

and using Proposition 2.1 again we get

$$\begin{aligned}\left\|\sum_{n\in B} \varepsilon_n e^*_{\rho(m)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| &\geqslant \left\|\sum_{n\in B} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &= \left\|x - \sum_{n\in A} e^*_n(x)e_n\right\| \\ &= \|x - \mathcal{G}_m(x)\|.\end{aligned}$$

If $B \cap A \neq \emptyset$, then

$$\begin{aligned}\left\|x - P_B(x)\right\| &= \left\|\sum_{n\in A\setminus B} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &\overset{(a)}{\geqslant} \left\|\sum_{n\in A\setminus B} \theta_n e^*_{\rho(m+1)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &\overset{(b)}{=} \left\|\sum_{n\in B\setminus A} \varepsilon_n e^*_{\rho(m+1)}(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &\overset{(c)}{\geqslant} \left\|\sum_{n\in B\setminus A} e^*_n(x)e_n + \sum_{n\notin A\cup B} e^*_n(x)e_n\right\| \\ &= \left\|\sum_{n\notin A} e^*_n(x)e_n\right\|,\end{aligned}$$

where $\theta_n = \pm 1$ have been chosen in such a way that $\operatorname{sgn}(\theta_n e^*_{\rho(m+1)}(x)) = \operatorname{sgn} e^*_n(x)$ for all $n \in A\setminus B$ and we picked $\varepsilon_n = \pm 1$ in order to satisfy $\operatorname{sgn}(\varepsilon_n e^*_{\rho(m+1)}(x)) = \operatorname{sgn} e^*_n(x)$ for all $n \in B\setminus A$. In (a) and (c) we used the fact that $K_s = 1$, and in (b) we used property (A). $\square$

From Example 3.3 we immediately deduce that the Haar system $(H_n^{(p)})_{n=1}^{\infty}$ is not a 1-greedy basis in $L_p[0, 1]$, $1 < p < \infty$.

Proposition 3.5. *Suppose $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space $(X, \|\cdot\|_X)$. For $1 \leqslant p < \infty$, let $Y = X \oplus_p \mathbb{R}$ endowed with the norm*

$$\|(x,\alpha)\|_Y = \left(\|x\|_X^p + |\alpha|^p\right)^{1/p}, \quad x \in X, \quad \alpha \in \mathbb{R}.$$

Denote $(y_n)_{n=0}^{\infty}$ the natural basis in Y : $\big((0,1), (e_1,0), (e_2,0), \dots\big)$. If (y_n) has property (A) *then (e_n) is isometrically isomorphic to the canonical ℓ_p-basis.*

Proof. Pick any $N \in \mathbb{N}$ and any linear combination $\sum_{n=1}^{N} \alpha_n e_n$. Without loss of generality we will assume that $|\alpha_1| \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_N|$. Then, using the fact that (y_n) has property (A), we have

$$\begin{aligned}\left\|\sum_{n=1}^{N} \alpha_n e_n\right\|_X &= \left\|\left(\sum_{n=1}^{N} \alpha_n e_n, 0\right)\right\|_Y \\ &= \left\|\alpha_1(0,1) + \sum_{n=2}^{N} \alpha_n e_n\right\|_Y \\ &= \left\|\left(\sum_{n=2}^{N} \alpha_n e_n, \alpha_1\right)\right\|_Y \\ &= \left(\left\|\sum_{n=2}^{N} \alpha_n e_n\right\|_X^p + |\alpha_1|^p\right)^{1/p}.\end{aligned}$$

Next we would play the same trick with the norm in X of $\sum_{n=2}^{N} \alpha_n e_n$. After N steps we would obtain

$$\left\|\sum_{n=1}^{N} \alpha_n e_n\right\|_X = \left(|\alpha_1|^p + \cdots + |\alpha_N|^p\right)^{1/p}. \qquad \square$$

The next two results can be shown in the same fashion and we omit their proof.

Proposition 3.6. *Let X be a Banach space with a basis $(x_n)_{n=1}^{\infty}$ and let $1 \leqslant p < \infty$. Consider the Banach space $Y = X \oplus_p \ell_p$ with the natural basis $(y_n)_{n=1}^{\infty} = \big((x_1,0), (0,e_1), (x_2,0), (0,e_2), \dots\big)$, where (e_n) denotes the unit vector basis of ℓ_p. If (y_n) has property* (A) *then (x_n) is isometrically equivalent to (e_n).*

Proposition 3.7. *Let $(X, \|\cdot\|)$ be a Banach space with a basis $(x_n)_{n=1}^{\infty}$. Consider the space $Y = X \oplus_1 X$ endowed with the norm*

$$\|(x_1,x_2)\|_Y = \|x_1\| + \|x_2\|.$$

The sequence $(y_n)_{n=1}^{\infty} = \big((x_1,0), (0,x_1), (x_2,0), (0,x_2), \dots\big)$ is a basis for Y. If (y_n) has property (A) *then (x_n) is isometrically equivalent to the canonical ℓ_1-basis.*

4. Renorming

In this section, we give partial results in connection with the open problems in §6. Suppose that (e_n) is a 1-greedy basis for a Banach space $(X, \|\cdot\|)$. By Theorem 1.1, (e_n) is unconditional with $K_s = 1$, and democratic with the democratic constant $= 1$. If we endow X with the equivalent *lattice norm*, defined for $x = \sum_{n=1}^{\infty} a_n e_n \in X$ by

$$\|x\|_\ell = \sup_{\theta_n = \pm 1} \left\| \sum_{n=1}^{\infty} \theta_n a_n e_n \right\|, \tag{5}$$

then (e_n) is unconditional in $(X, \|\cdot\|_\ell)$ with $K_{\rm u} = 1$, but one could expect the democratic constant of (e_n) in the new norm to increase. This is not the case and (e_n) remains 1-superdemocratic in $(X, \|\cdot\|_\ell)$. Indeed, for any $n \in \mathbb{N}$ and any $A \subset \mathbb{N}$ with $|A| = n$, taking into account the 1-superdemocracy of (e_n) in $(X, \|\cdot\|)$, we have

$$\left\| \sum_{k \in A} \theta_k e_k \right\|_\ell = \sup_{\varepsilon_k = \pm 1} \left\| \sum_{k \in A} \varepsilon_k \theta_k e_k \right\| = \left\| \sum_{k \in A} e_k \right\|,$$

for any $(\theta_k)_{k \in A}$ signs. Actually we will show that (e_n) still is 1-greedy after renorming X with the norm in (5). In its proof we will use the following elementary lemma.

Lemma 4.1. *Let (e_n) be an unconditional basis for a Banach space X. Then, for each $x = \sum_{n=1}^{\infty} a_n e_n \in X$ there exists a sequence of signs (θ_n) (which depends on x) so that*

$$\|x\|_\ell = \left\| \sum_{n=1}^{\infty} \theta_n a_n e_n \right\|.$$

Proof. It is easy to see that the map from the topological product space $\{-1, 1\}^{\mathbb{N}}$ into X which assigns to each sequence of signs (θ_n) the vector $\sum_{n=1}^{\infty} \theta_n a_n e_n$ is continuous. Composing with the norm in X gives us a continuous map from $\{-1, 1\}^{\mathbb{N}}$ into $\mathbb{R}$:

$$(\theta_n) \mapsto \left\| \sum_{n=1}^{\infty} \theta_n a_n e_n \right\|.$$

By compactness, there is a choice of signs (θ_n) where this map attains its maximum. □

Proposition 4.2. *Let (e_n) be a 1-greedy basis for the Banach space $(X, \|\cdot\|)$. Then (e_n) is (1-unconditional and) 1-greedy in $(X, \|\cdot\|_\ell)$.*

Proof. Take any $x = \sum_{k=1}^{\infty} a_n e_n \in X$. Without loss of generality we assume that the coefficients of x in absolute value are non-increasing (otherwise we work with the greedy ordering of x). Thus for each $m \in \mathbb{N}$,

$$\|x - \mathcal{G}_m(x)\|_\ell = \left\| \sum_{n=m+1}^{\infty} a_n e_n \right\|_\ell = \sup_{\pm 1} \left\| \sum_{n=m+1}^{\infty} \pm a_n e_n \right\| = \left\| \sum_{n=m+1}^{\infty} \theta_n a_n e_n \right\|, \tag{6}$$

where (θ_n) is the sequence of signs given by the previous lemma. Put

$$y = \sum_{n=1}^{\infty} \theta_n a_n e_n,$$

where $\theta_1 = \theta_2 = \cdots = \theta_m = 1$. Then $\mathcal{G}_m(y) = \sum_{n=1}^{m} a_n e_n$ and, since (e_m) is 1-greedy in $(X, \|\cdot\|)$, we have

$$\left\| \sum_{n=m+1}^{\infty} \theta_n a_n e_n \right\| = \|y - \mathcal{G}_m(y)\| \leqslant \sigma_m^{\|\cdot\|}(y). \tag{7}$$

Now, for each set $B \subset \mathbb{N}$ of cardinality m,

$$\sigma_m^{\|\cdot\|}(y) \leqslant \left\| y - \sum_{k\in B} \theta_k a_k e_k \right\| \leqslant \left\| \sum_{n=1}^{\infty} a_n e_n - \sum_{k\in B} a_k e_k \right\|_{\ell},$$

which implies

$$\sigma_m^{\|\cdot\|}(y) \leqslant \min \left\{ \left\| \sum_{n=1}^{\infty} a_n e_n - \sum_{k\in B} a_k e_k \right\|_{\ell} : B \subset \mathbb{N}, |B| = m \right\} = \sigma_m^{\|\cdot\|_{\ell}}(x). \tag{8}$$

Combining (6), (7) and (8) we obtain

$$\|x - \mathcal{G}_m(x)\|_{\ell} \leqslant \sigma_m^{\|\cdot\|_{\ell}}(x),$$

i.e., (e_n) is 1-greedy in $(X, \|\cdot\|_{\ell})$. □

Analogously, if (e_n) is C-greedy in $(X, \|\cdot\|)$ and we equivalently renorm X with the lattice norm, then one may argue as above to show that, in fact, (e_n) is 1-unconditional and C-greedy in $(X, \|\cdot\|_{\ell})$.

A basic tool to analyze unconditional bases in $L_p[0, 1]$ for $1 < p < \infty$ is provided by the following consequence of Khintchine's inequalities.

Proposition 4.3. *Let $1 < p < \infty$. If $(\psi_n)_{n=1}^{\infty}$ is an unconditional basis for $(L_p[0, 1], \|\cdot\|_p)$ with biorthogonal functionals (ψ_n^*), then the expression*

$$|||f||| = \left(\int_0^1 \left(\sum_{n=1}^{\infty} |\psi_n^*(f)|^2 |\psi_n(t)|^2 \right)^{p/2} dt \right)^{1/p}, \quad f \in L_p[0, 1],$$

gives a norm on $L_p[0, 1]$ which is equivalent to the standard L_p-norm.

Hence, as a particular case of the above proposition, one obtains:

Proposition 4.4. *For each* $1 < p < \infty$ *there exists a constant* $C = C(p)$ *so that*

$$C^{-1}\left(\int_0^1\left(\sum_{n=1}^{\infty}|a_n|^2|H_n^{(p)}(t)|^2\right)^{p/2}dt\right)^{1/p}$$
$$\leqslant\left\|\sum_{n=1}^{\infty}a_nH_n^{(p)}\right\|_{L_p}\leqslant C\left(\int_0^1\left(\sum_{n=1}^{\infty}|a_n|^2|H_n^{(p)}(t)|^2\right)^{p/2}dt\right)^{1/p},$$

for any sequence $(a_n)\in c_{00}$.

Proof. Given $f=\sum_{n=1}^{\infty}a_nH_n^{(p)}\in L_p[0,1]$ $(1<p<\infty)$, put

$$|||f|||=\left(\int_0^1\left(\sum_{n=1}^{\infty}|a_n|^2|H_n^{(p)}(t)|^2\right)^{p/2}dt\right)^{1/p} \tag{9}$$

and appeal to Proposition 4.3. □

Sometimes it is convenient to describe the normalized Haar basis in $L_p[0,1]$ as a sequence of "layers" as follows. Let h_0^0 be the constant function 1. For $n\geqslant 0$ and $1\leqslant k\leqslant 2^n$ we define h_k^n thus:

$$h_k^n(t)=\begin{cases}2^{n/p} & \text{if } t\in[\frac{2k-2}{2^{n+1}},\frac{2k-1}{2^{n+1}}),\\ -2^{n/p} & \text{if } t\in[\frac{2k-1}{2^{n+1}},\frac{2k}{2^{n+1}}),\\ 0 & \text{otherwise.}\end{cases}$$

Our next result proves that $L_p[0,1]$ can be equivalently renormed so that some subbasis of the Haar system is 1-unconditional and $(1+\varepsilon)$-democratic for the new norm. Unfortunately we are unable to prove it for the whole basis and cannot get rid of ε (see Problem 6.2).

Proposition 4.5. *Let* $1<p<\infty$. *For each* $\varepsilon>0$ *there exists an increasing sequence* $(n_i)_{i=1}^{\infty}$ *of non-negative integers such that the "lacunary Haar system"* $\left((h_j^{n_i})_{j=1}^{2^{n_i}}\right)_{i=0}^{\infty}$ *is* 1-*unconditional*, $(1+\varepsilon)$-*democratic in* $(L_p[0,1],|||\cdot|||)$ *and the closed linear span of* $\left((h_j^{n_i})_{j=1}^{2^{n_i}}\right)_{i=0}^{\infty}$ *in* $(L_p[0,1],|||\cdot|||)$ *is isomorphic to* $(L_p[0,1],\|\cdot\|_p)$.

Proof. The proof relies basically on an idea that appeared in [5]. Given $\varepsilon>0$, pick $\alpha\in\mathbb{N}$ such that

$$\frac{2^{\alpha/p}}{(2^{2\alpha/p}-1)^{1/2}}\cdot\frac{2^{\alpha/p}}{(2^{\alpha}-1)^{1/p}}\leqslant 1+\varepsilon.$$

Consider the sequence (n_i) defined by $n_0=0$ and $n_{i+1}=n_i+\alpha$ and the subbasis of the Haar system

$$\mathcal{S}_{\varepsilon}=\left((h_j^{n_i})_{j=1}^{2^{n_i}}\right)_{i=0}^{\infty}=(h_0^0,\underbrace{h_1^{n_1},h_2^{n_1},\ldots,h_{2^{n_1}}^{n_1}}_{n_1^{\text{th}}\text{-layer}},\underbrace{h_1^{n_2},h_2^{n_2},\ldots,h_{2^{n_2}}^{n_2}}_{n_2^{\text{th}}\text{-layer}},\ldots).$$

Note that for each $t \in [0, 1)$, the non-zero values of the functions $|h_k^{n_i}(t)|^p$, $i = 1, 2, \ldots$ belong to a geometric progression of ratio 2^α. If A is any finite subset of $\mathcal{S}_\varepsilon$, put

$$M(t) = \max \left\{ n_i : t \in \operatorname{supp} h_k^{n_i}, h_k^{n_i} \in A \right\},$$

and let $M(t) = -\infty$ if $t \notin \bigcup_{h_k^{n_i} \in A} h_k^{n_i}$. Thus for each $t \in [0, 1)$ we see that

$$\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p \leqslant 2^{M(t)} \sum_{i=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^i = \frac{2^\alpha}{2^\alpha - 1} \cdot 2^{M(t)},$$

hence

$$2^{M(t)} \geqslant \frac{2^\alpha - 1}{2^\alpha} \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p.$$

Now,

$$\begin{aligned} \int_0^1 \left(\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \right)^{p/2} dt &\geqslant \int_0^1 2^{M(t)}\, dt \\ &\geqslant \frac{2^\alpha - 1}{2^\alpha} \int_0^1 \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p\, dt \\ &= \frac{2^\alpha - 1}{2^\alpha} |A|. \end{aligned}$$

Therefore, we obtain

$$\left|\left|\left| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \right|\right|\right| \geqslant \left(\frac{2^\alpha - 1}{2^\alpha} \right)^{1/p} |A|^{1/p}. \tag{10}$$

On the other hand, for each $t \in [0, 1)$ we have

$$\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \leqslant \left(2^{\frac{M(t)}{p}} \right)^2 \sum_{j=0}^{\infty} \left(\frac{1}{2^{2\alpha/p}} \right)^j = 2^{\frac{2M(t)}{p}} \frac{2^{2\alpha/p}}{2^{2\alpha/p} - 1}.$$

Then,

$$\begin{aligned} \int_0^1 \left(\sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^2 \right)^{p/2} &\leqslant \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} \int_0^1 2^{M(t)}\, dt \\ &\leqslant \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} \int_0^1 \sum_{\{h_k^{n_i} \in A\}} |h_k^{n_i}(t)|^p\, dt \\ &= \frac{2^\alpha}{(2^{2\alpha/p} - 1)^{p/2}} |A|. \end{aligned}$$

Thus we obtain

$$\left|\!\left|\!\left| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \right|\!\right|\!\right| \leqslant \frac{2^{\alpha/p}}{(2^{2\alpha/p}-1)^{1/2}} |A|^{1/p}. \tag{11}$$

So given any other set $B \subset \mathcal{S}_\varepsilon$ such that $|B| = |A|$, Eqs. (10) and (11) yield

$$\begin{aligned}\left|\!\left|\!\left| \sum_{\{h_m^{n_j} \in B\}} h_m^{n_j} \right|\!\right|\!\right|^p &\leqslant \frac{2^{\alpha}}{(2^{2\alpha/p}-1)^{p/2}} |B| \\ &\leqslant \frac{2^{\alpha}}{(2^{2\alpha/p}-1)^{p/2}} \frac{2^{\alpha}}{2^{\alpha/p}-1} \left|\!\left|\!\left| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \right|\!\right|\!\right|^p \\ &\leqslant (1+\varepsilon)^p \left|\!\left|\!\left| \sum_{\{h_k^{n_i} \in A\}} h_k^{n_i} \right|\!\right|\!\right|^p .\end{aligned}$$

The last statement of the proposition follows from Gamlen and Gaudet's theorem [3] and from the equivalence of norms given by Proposition 4.4. □

5. Examples

If $(e_n)_{n=1}^{\eta}$ is 1-greedy, by Theorem 1.1, $(e_n)_{n=1}^{\eta}$ is 1-democratic and unconditional with $K_s = 1$. Our first example shows that $(e_n)_{n=1}^{\eta}$ need not be 1-superdemocratic, and hence Proposition 3.2 fails when the space is finite-dimensional. In particular, it shows that a 1-greedy basis need not be 1-unconditional (at least in a two-dimensional space!).

Example 5.1. Put

$$\mathcal{B} = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1, xy \geqslant 0\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1, xy \leqslant 0\right\},$$

and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$, i.e., for each $x \in X$

$$\|x\|_{\mathcal{B}} = \inf\left\{t > 0 : \frac{x}{t} \in \mathcal{B}\right\}.$$

$X = (\mathbb{R}^2, \|\cdot\|_{\mathcal{B}})$ is a Banach space and the unit vectors $e_1 = (1, 0)$, $e_2 = (0, 1)$ are a basis for X. It is immediate to check that $\|P_{\{i\}}\| \leqslant 1$ for $i = 1, 2$, hence by Proposition 2.2, (e_1, e_2) is 1-greedy.

On the other hand, (e_1, e_2) is not 1-superdemocratic since $\|e_1+e_2\| = \sqrt{2}$ whereas $\|e_1-e_2\| = 2$. Therefore, (e_1, e_2) cannot be 1-unconditional.

One might think, in view of Example 5.1, that a basis which is 1-greedy, 1-superdemocratic and such that $K_s = 1$ would be 1-unconditional. This is not the case as the next two-dimensional example shows.

Example 5.2. Let

$$A_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1, x \geqslant 0, y \geqslant 0, x \leqslant y\},$$
$$A_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1, x \leqslant 0, y \geqslant 0, |x| \geqslant y\},$$
$$A_3 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1, x \leqslant 0, y \leqslant 0, |x| \leqslant |y|\},$$
$$A_4 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1, x \geqslant 0, y \leqslant 0, x \geqslant |y|\},$$

and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Now, take $\mathcal{B}$ the convex hull of A and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$. $(X, \|\cdot\|_{\mathcal{B}})$ is a Banach space, of which the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are a basis. Clearly, $\|P_{\{i\}}\| \leqslant 1$ for $i = 1, 2$, hence by Proposition 2.2, (e_1, e_2) is 1-greedy. It is also immediate that $\|\theta_1 e_1 + \theta_2 e_2\| = \|\varepsilon_1 e_1 + \varepsilon_2 e_2\|$ for any choices of signs $\{\theta_i\}_{i=1}^2$ and $\{\varepsilon_i\}_{i=1}^2$, therefore the basis is 1-superdemocratic. Nevertheless, (e_1, e_2) is not 1-unconditional since, for instance, given any $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2})$ the vector $x = (\cos\alpha, \sin\alpha)$ has norm $= 1$ whereas the vector $x' = (\cos\alpha, -\sin\alpha)$ has norm strictly bigger than 1.

By Theorem 1.1, a 1-unconditional and 1-democratic basis is greedy with greedy constant at most 2. Can we do any better? The next example gives a basis in an infinite-dimensional Banach space which is 1-unconditional and 1-superdemocratic but not 1-greedy.

Example 5.3. Let X be the set of all real sequences $x = (x_1, x_2, \dots) \in \ell_2$ such that

$$\|x\|_1 = \sum_{n=1}^{\infty} \frac{|x_n|}{\sqrt{n}}$$

is finite. Taking into account (we will see below why) that the inequality

$$\frac{1}{2} \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \leqslant \sqrt{N} \tag{12}$$

holds for all $N \in \mathbb{N}$, we define on X the norm given by:

$$\|x\| = \max\left\{\|x\|_{\ell_2}, \tfrac{1}{2}\|x\|_1\right\}.$$

Then $(X, \|\cdot\|)$ is a Banach space. Let $e_n \in X$, $n = 1, 2, \dots$, be the vector whose kth coordinate is 1 if $n = k$ and 0 otherwise. Denote by X_0 the subspace of X generated by $(e_n)_{n=1}^{\infty}$.

It is easy to see that (e_n) is a 1-unconditional basis for X_0.

On the other hand, given any subset $A \subset \mathbb{N}$, we have

$$\left\|\sum_{k\in A} e_k\right\|_1 \leqslant \left\|\sum_{k=1}^{|A|} e_k\right\|_1 = \sum_{k=1}^{|A|} \frac{1}{\sqrt{k}},$$

which implies, using (12), that

$$\left\|\sum_{k\in A} e_k\right\| = \left\|\sum_{k\in A} e_k\right\|_{\ell_2} = |A|^{1/2},$$

hence (e_n) is 1-democratic. In fact, (e_n) is 1-superdemocratic.

Let us show that (e_n) does not have property (A). Pick $n \in \mathbb{N}$ such that

$$\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) > \sqrt{1+\frac{1}{2}+\cdots+\frac{1}{n}}. \tag{13}$$

Then,

$$\left\|\left(1,\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{n}},0,\ldots\right)\right\| = \frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right),$$

whereas

$$\begin{aligned}&\left\|\left(0,\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{n}},1,0,\ldots\right)\right\| \\ &\quad = \max\left\{\frac{1}{2}\left(\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{\sqrt{n+1}}\right), \sqrt{1+\frac{1}{2}+\cdots+\frac{1}{n}}\right\} \\ &\quad \neq \left\|\left(1,\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{n}},0,\ldots\right)\right\|.\end{aligned}$$

Let us recall that a basis $(e_n)_{n=1}^{\infty}$ is *subsymmetric* if it is unconditional and for every increasing sequence of integers $\{n_i\}_{i=1}^{\infty}$, the subbasis $(e_{n_i})_{i=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$. The *subsymmetry constant* of (e_n) is the smallest constant $C \geqslant 1$ such that given any scalars $(a_i) \in c_{00}$, we have

$$\left\|\sum_{i=1}^{\infty} \theta_i a_i e_{n_i}\right\| \leqslant C \left\|\sum_{i=1}^{\infty} a_i e_i\right\|$$

for all sequences of signs (θ_i) and all increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$. In this case we say that (e_n) is *C-subsymmetric*.

Since a 1-subsymmetric basis (e_n) is 1-unconditional and 1-democratic, by Theorem 1.1 it follows that (e_n) is greedy with greedy constant $\leqslant 2$.

The following example, in combination with Theorem 3.4, shows that a 1-subsymmetric basis need not be 1-greedy. It is interesting to point out here that this was precisely the first counterexample that proved that a subsymmetric basis need not be symmetric (see [4]).

Example 5.4. Let $(X, \|\cdot\|)$ be the Banach space of all sequences of scalars $x = (x_1, x_2, \ldots)$ for which

$$\|x\| = \sup \sum_{i=1}^{\infty} \frac{|x_{n_i}|}{\sqrt{i}} < \infty,$$

the supremum being taken over all increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$.

The unit vectors (e_i) form a 1-subsymmetric basis of X, but (e_i) fails to be 1-greedy because it does not have property (A). Indeed, take $x = (1, \frac{1}{\sqrt{2}}, 0, 0\cdots)$ and, for instance, the greedy permutation of x given by $\pi(1) = 3, \pi(2) = 2$. Then, $\|(1, \frac{1}{\sqrt{2}}, 0, 0\cdots)\| = 1 + \frac{1}{2}$ whereas $\|(0, \frac{1}{\sqrt{2}}, 1, 0\cdots)\| = \sqrt{2}$.

Example 5.5 (*Greedy does not imply subsymmetric*). It was proved in [8] that for $1 < p < \infty$, $(H_n^{(p)})_{n=1}^{\infty}$ is a greedy basis in $L_p[0,1]$ with a greedy constant strictly bigger than 1 (unless for $p = 2$ that the greedy constant is $= 1$). Clearly $(H_n^{(p)})_{n=1}^{\infty}$ is not subsymmetric since if we consider $n_k = 2^{k+1} - 1$, $k = 1, 2, \ldots$, then the subbasis $(H_{n_k}^{(p)})_{k=1}^{\infty}$ is isometrically isomorphic to ℓ_p, which is not isomorphic to $L_p[0, 1]$.

Now we shall present two examples which show that a 1-greedy basis need not be 1-symmetric. The first one, essentially due to the referee, is a nice an simple way to define a norm $|||\cdot|||$ on c_0 equivalent to the standard one, starting with a two-dimensional norm, so that the canonical basis $(e_n)_{n=1}^{\infty}$ is 1-greedy but not 1-symmetric in the new norm. Unfortunately, though, the sequence $(e_n)_{n\geqslant 2}$ is 1-symmetric with respect to $|||\cdot|||$!

Example 5.6. Consider the following two-dimensional norm:

$$\|(x, y)\| = \max\left\{|x|, |y|, \tfrac{5}{6}|x| + \tfrac{1}{3}|y|\right\}.$$

Now endow c_0 with the norm

$$|||(a_n)_{n=1}^{\infty}||| = \max\left\{\sup_{1\leqslant i<j} \|(a_i, a_j)\|, \sup_{2\leqslant i<j} \|(a_j, a_i)\|\right\}.$$

It is immediate to see that $|||\cdot|||$ is equivalent to $\|(a_n)_{n=1}^{\infty}\|_\infty = \sup_n |a_n|$. One can also readily check that the standard unit vector basis of $(c_0, |||\cdot|||)$ is 1-unconditional and has property (A), hence it is 1-greedy by Theorem 3.4. But it cannot be 1-symmetric since

$$|||(\tfrac{3}{4}, \tfrac{1}{2}, 0, 0, \ldots)||| = \tfrac{19}{24}$$

whereas

$$|||(\tfrac{1}{2}, \tfrac{3}{4}, 0, 0, \ldots)||| = \tfrac{3}{4}.$$

The other example is more involved and finite-dimensional in nature. It gives for each $n \in \mathbb{N}$ an n-dimensional Banach space (very close to a Hilbert space) with a 1-greedy basis whose symmetry constant approaches 1 as n tends to ∞. We are still unable to find a sequence $(X_n)_{n=1}^{\infty}$ of spaces with $\dim X_n = n$, so that each X_n has a 1-greedy basis whose symmetry constant goes to ∞ as n increases.

Example 5.7. We are going to construct the unit ball of an n-dimensional Banach space as follows. For each $i = 1, 2, \ldots, n$, let $\mathcal{E}_i$ denote the Euclidean unit ball in the hyperplane $H_i = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0\}$, i.e.,

$$\mathcal{E}_i = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ and } \|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \leqslant 1\right\},$$

and let $\mathcal{E}$ be the Euclidean unit ball in $\mathbb{R}^n$. We define the set $\mathcal{A}$ to be

$$\mathcal{A} = \left\{a = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0 \text{ and } \sum_{j=1}^{n} |x_j|^2 \leqslant 1\right\}.$$

Now, for each different choice of signs $\theta^{(j)} = (\theta_1^{(j)}, \dots, \theta_n^{(j)})$, $j = 1, \dots, 2^n$, put

$$\mathcal{A}_j = \left\{ a = (\theta_1^{(j)} x_1, \dots, \theta_n^{(j)} x_n) : (x_i)_{i=1}^n \in \mathcal{A} \right\}.$$

Let us observe that all of the sets $\mathcal{E}_i$'s and $\mathcal{A}_j$'s are convex. Finally, put

$$\mathcal{S} = \left(\bigcup_{i=1}^{n} \mathcal{E}_i \right) \cup \left(\bigcup_{j=1}^{2^n} \mathcal{A}_j \right).$$

Let

$$\mathcal{B} = \mathrm{co}(\mathcal{S}),$$

the convex hull of $\mathcal{S}$, and let $\|\cdot\|_{\mathcal{B}}$ denote the Minkowski functional of $\mathcal{B}$. $X = (\mathbb{R}^n, \|\cdot\|_{\mathcal{B}})$ is a Banach space. We will prove that the unit vector basis $(e_i)_{i=1}^n$ of X is 1-greedy but it is not 1-symmetric. First we make a few geometric remarks that we will be using in the sequel. Note that $\mathcal{B}$ consists of all sums $\sum_{j=1}^{2^n} \lambda_j a_j + \sum_{i=1}^n \mu_i b_i$ such that $a_j \in \mathcal{A}_j$ for $j = 1, \dots, 2^n$, $b_i \in \mathcal{E}_i$ for $i = 1, \dots, n$ and $\lambda_1, \dots, \lambda_{2^n}, \mu_1, \dots, \mu_n$ are non-negative real numbers satisfying $\sum_{j=1}^{2^n} \lambda_j + \sum_{i=1}^n \mu_i \leqslant 1$. It then follows that for $x \in \mathbb{R}^n$

$$\|x\|_{\mathcal{B}} = \inf \left\{ \sum_{j=1}^{2^n} \lambda_j + \sum_{i=1}^{n} \mu_i : \lambda_j \geqslant 0, \mu_i \geqslant 0 \text{ and } x = \sum_{j=1}^{2^n} \lambda_j a_j + \sum_{i=1}^{n} \mu_i b_i \right\}. \tag{14}$$

Moreover, it is an easy consequence of the compactness of the sets $\mathcal{A}_j$ and $\mathcal{E}_i$ that this infimum is attained.

Let us see that $(e_k)_{k=1}^n$ is 1-unconditional. Given $\varepsilon = (\varepsilon_k)_{k=1}^n$, where $\varepsilon_k = \pm 1$, let T_ε be the map $\sum_{k=1}^n x_k e_k \to \sum_{k=1}^n \varepsilon_k x_k e_k$ from X to X. Each such map T_ε is linear. We must prove that $\sup_\varepsilon \|T_\varepsilon\| \leqslant 1$. Since $\mathcal{B} = \mathrm{co}(\mathcal{S})$, it suffices to show that $T_\varepsilon(\mathcal{S}) \subset \mathcal{S}$ for each ε. But if $x \in \mathcal{S}$, then either $x \in \mathcal{E}_i$ for some i, in which case $T_\varepsilon(x) \in \mathcal{E}_i \subset \mathcal{S}$, or $x \in \mathcal{A}_j$ for some $1 \leqslant j \leqslant 2^n$, in which case $T_\varepsilon(x) \in \mathcal{A}_{j'} \subset \mathcal{S}$.

To check that $(e_k)_{k=1}^n$ has property (A), let us pick $x = (x_1, \dots, x_n)$ a vector of the unit ball of X so that at least one of its coordinates is zero (otherwise there is nothing to prove). That is, x belongs to the intersection of $\mathcal{B}$ with at least one hyperplane H_i. Since $\mathcal{S} \subset \mathcal{E}$, it follows that $\mathcal{B}$, the convex hull of $\mathcal{S}$, is also contained in $\mathcal{E}$, hence $\mathcal{B} \cap H_i \subset \mathcal{E} \cap H_i = \mathcal{E}_i$. We conclude that $\mathcal{B} \cap H_i = \mathcal{E}_i$. We will prove that if $x \in \mathcal{E}_i$ then $\|x\|_2 = \|x\|_{\mathcal{B}}$. It may be assumed that $\|x\|_2 = 1$. Taking into account the expression of $\|\cdot\|_{\mathcal{B}}$ given in (14) and the fact that $x \in \mathcal{E}_i$, we deduce that $\|x\|_{\mathcal{B}} \leqslant 1$. Suppose that $\|x\|_{\mathcal{B}} < 1$. Pick a representation of x,

$$x = \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^{n} \bar{\mu}_i \bar{b}_i$$

such that $\bar{\lambda}_j \geqslant 0$, $\bar{\mu}_i \geqslant 0$, $\bar{a}_j \in A_j$, $\bar{b}_i \in \mathcal{E}_i$ and

$$\|x\|_{\mathcal{B}} = \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^{n} \bar{\mu}_i.$$

Hence we would have

$$
\begin{aligned}
1 = \|x\|_2 &= \left\| \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^{n} \bar{\mu}_i \bar{b}_i \right\| \\
&\leqslant \sum_{j=1}^{2^n} \bar{\lambda}_j \|\bar{a}_j\|_2 + \sum_{i=1}^{n} \bar{\mu}_i \|\bar{b}_i\|_2 \\
&\leqslant \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^{n} \bar{\mu}_i \\
&< 1.
\end{aligned}
$$

Therefore, to evaluate the $\|\cdot\|_{\mathcal{B}}$-norm of an element of $\mathcal{E}_i$ we can use its $\|\cdot\|_2$-norm. Now, if π is a greedy permutation of x, the vector $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ belongs to $\mathcal{E}_k$ for some $1 \leqslant k \leqslant n$ and

$$\|x_\pi\|_{\mathcal{B}} = \|x_\pi\|_2 = \|x\|_2 = \|x\|_{\mathcal{B}}.$$

By Theorem 3.4, $(e_k)_{k=1}^n$ is 1-greedy.

It remains to be proved that $(e_k)_{k=1}^n$ is not 1-symmetric. We will see that there exist vectors $x = (x_1, \ldots, x_n) \in X$ with $\|x\|_{\mathcal{B}} = 1$ such that for some permutation π, the norm of $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ is strictly bigger than 1. Let us take $x = (x_1, \ldots, x_n) \in \mathcal{A}$ such that $x_1 > x_2 > \cdots > x_n > 0$ and $\|x\|_2 = 1$. Since $\mathcal{A} \subset \mathcal{B} \subset \mathcal{E}$, it follows that

$$1 = \|x\|_2 \leqslant \|x\|_{\mathcal{B}} \leqslant 1,$$

hence $\|x\|_{\mathcal{B}} = 1$.
Now consider $x' = (x_n, x_{n-1}, \ldots, x_2, x_1)$. Obviously, $\|x'\|_2 = 1$. We aim to show that $\|x\|_{\mathcal{B}} > 1$. Suppose the contrary. Then, since $\|x'\|_{\mathcal{B}}$ cannot be strictly less than 1, the only option is $\|x'\|_{\mathcal{B}} = 1$.

We choose a representation of x' where its $\|\cdot\|_{\mathcal{B}}$-norm is attained,

$$x' = \sum_{j=1}^{2^n} \bar{\lambda}_j \bar{a}_j + \sum_{i=1}^{n} \bar{\mu}_i \bar{b}_i \quad \text{and} \quad 1 = \sum_{j=1}^{2^n} \bar{\lambda}_j + \sum_{i=1}^{n} \bar{\mu}_i.$$

Clearly, in the above representation it must be $\|\bar{a}_j\|_2 = 1 = \|\bar{b}_i\|_2$ for all $j = 1, \ldots, 2^n$ and all $i = 1, \ldots, n$. This way we have a vector in the Euclidean unit sphere of $\mathbb{R}^n$ written down as a convex combination of vectors in the Euclidean unit sphere of $\mathbb{R}^n$ as well. Using the strict convexity (or rotundity) of $\mathcal{E}$ we infer that $\bar{a}_j = \bar{b}_i = x'$, which is impossible by our choice of x'.

Let us note that as n grows larger the basis becomes more and more symmetric. Let $x = (x_1, \ldots, x_n) \in X$ such that $\|x\|_{\mathcal{B}} = 1$. At least one of the coordinates of x, say x_n, is $\leqslant \frac{1}{\sqrt{n}}$. Then, given any permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we have

$$
\begin{aligned}
\|x_1 e_{\pi(1)} + \cdots + x_n e_{\pi(n)}\|_{\mathcal{B}} &\leqslant \|x_1 e_{\pi(1)} + \cdots + x_{n-1} e_{\pi(n-1)}\|_{\mathcal{B}} + \|x_n e_{\pi(n)}\|_{\mathcal{B}} \\
&\leqslant \|x_1 e_1 + \cdots + x_{n-1} e_{n-1}\|_{\mathcal{B}} + \frac{1}{\sqrt{n}} \\
&\leqslant \|x\|_{\mathcal{B}} + \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{n}}.
\end{aligned}
$$

6. Final remarks

It is clear that our work leaves a lot or open questions. Clearly, the most important is:

Problem 6.1. Does there exist a 1-greedy basis $(e_n)_{n=1}^{\infty}$ which is not symmetric in an infinite-dimensional Banach space X?

The natural approach to tackle this question is to use renorming. So, in particular one may ask: suppose $(e_n)_{n=1}^{\infty}$ is a greedy basis in a Banach space X. Does there exist an equivalent norm $|||\cdot|||$ on X so that $(e_n)_{n=1}^{\infty}$ is 1-greedy in $(X, |||\cdot|||)$? Interestingly enough, it is not the case. It was recently shown by T. Schlumprecht (unpublished) that neither Johnson–Figiel–Tsirelson type space (see [2, Section 2]) nor H_1 can be renormed so that their natural bases are 1-greedy.

However, the following are still open:

Problem 6.2. Can we equivalently renorm $(L_p[0,1], \|\cdot\|_p)$ $(1 < p < \infty)$ so that the normalized Haar system is 1-greedy in the new norm?

Problem 6.3. Suppose $(e_n)_{n=1}^{\infty}$ is a (super)democratic basis in a Banach space X. Does there exist an equivalent norm $|||\cdot|||$ on X so that $(e_n)_{n=1}^{\infty}$ is 1-(super)democratic in $(X, |||\cdot|||)$?

A different problem is suggested by Theorem 1.1, Proposition 4.2, and Examples 5.1 and 5.2:

Problem 6.4. Does 1-greedy imply 1-unconditional in an infinite-dimensional Banach space?

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